

BRAUER GROUPS OF QUOT SCHEMES

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ABSTRACT. Let X be an irreducible smooth complex projective curve. Let $\mathcal{Q}(r, d)$ be the Quot scheme parametrizing all coherent subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-d$. There are natural morphisms $\mathcal{Q}(r, d) \rightarrow \mathrm{Sym}^d(X)$ and $\mathrm{Sym}^d(X) \rightarrow \mathrm{Pic}^d(X)$. We prove that both these morphisms induce isomorphism of Brauer groups if $d \geq 2$. Consequently, the Brauer group of $\mathcal{Q}(r, d)$ is identified with the Brauer group of $\mathrm{Pic}^d(X)$ if $d \geq 2$.

1. INTRODUCTION

Let X be an irreducible smooth projective curve defined over \mathbb{C} . For any $r \geq 1$, consider the holomorphic trivial vector bundle $\mathcal{O}_X^{\oplus r}$ on X . For any $d \geq 0$, let $\mathcal{Q}(r, d)$ denote the Quot scheme that parametrizes all torsion quotients of degree d of the \mathcal{O}_X -module $\mathcal{O}_X^{\oplus r}$. This $\mathcal{Q}(r, d)$ is an irreducible smooth complex projective variety of dimension rd .

For every $Q \in \mathcal{Q}(r, d)$, we have a corresponding short exact sequence

$$0 \rightarrow \mathcal{F}(Q) \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow Q \rightarrow 0.$$

Sending Q to the scheme theoretic support of Q , we get a morphism

$$\varphi : \mathcal{Q}(r, d) \rightarrow \mathrm{Sym}^d(X).$$

Sending any $Q \in \mathcal{Q}(r, d)$ to the holomorphic line bundle $\bigwedge^r \mathcal{F}(Q)^*$, we get a morphism

$$\varphi' : \mathcal{Q}(r, d) \rightarrow \mathrm{Pic}^d(X).$$

On the other hand, we have the morphism

$$\xi_d : \mathrm{Sym}^d(X) \rightarrow \mathrm{Pic}^d(X)$$

that sends any (x_1, \dots, x_d) to the holomorphic line bundle $\mathcal{O}_X(\sum_{i=1}^d x_i)$. Note that $\varphi' = \xi_d \circ \varphi$.

The Brauer group of a smooth complex projective variety M will be denoted by $\mathrm{Br}(M)$.

Our aim here is to prove the following:

Theorem 1.1. *For the morphisms φ and ξ_d , the pullback homomorphisms of Brauer groups*

$$\varphi^* : \mathrm{Br}(\mathrm{Sym}^d(X)) \rightarrow \mathrm{Br}(\mathcal{Q}(r, d)) \quad \text{and} \quad \xi_d^* : \mathrm{Br}(\mathrm{Pic}^d(X)) \rightarrow \mathrm{Br}(\mathrm{Sym}^d(X))$$

are isomorphisms provided $d \geq 2$.

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Theorem 1.1 is proved in Lemma 4.2 and Lemma 6.1.

Fixing a point $x_0 \in X$, construct an embedding

$$\delta : \mathcal{Q}(r, d) \longrightarrow \mathcal{Q}(r, d + r)$$

by sending any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ to $\mathcal{F} \otimes \mathcal{O}_X(-x_0)$.

The following is proved in Corollary 6.2:

Proposition 1.2. *The pullback homomorphism for δ*

$$\delta^* : \text{Br}(\mathcal{Q}(r, d + r)) \longrightarrow \text{Br}(\mathcal{Q}(r, d))$$

is an isomorphism if $g \geq 2$.

Now assume that $r, \text{genus}(X) \geq 2$; if $r = 2$, then assume that $\text{genus}(X) \geq 3$. Iterating the morphism δ , we get an ind scheme; this ind scheme is the projectivization of a vector bundle over the moduli stack of vector bundles on X of rank r [BGL]. Using this, from Proposition 1.2 and Theorem 1.1, it can be deduced that the Brauer group of the moduli stack of vector bundles on X of rank r and fixed determinant vanishes. This result was proved earlier in [BH]. Using this vanishing result it can be deduced that the Brauer group of the moduli space of stable vector bundles on X of rank r and fixed determinant of degree d is a cyclic group of order $\text{g.c.d.}(r, d)$. This result was proved earlier in [BBGN].

2. BRAUER GROUP

Let M be an irreducible smooth projective variety defined over \mathbb{C} . Let \mathcal{O}_M^* denote the multiplicative sheaf on M of holomorphic functions with values in $\mathbb{C} \setminus \{0\}$. The *cohomological Brauer group* $\text{Br}'(M)$ is the torsion subgroup of the cohomology group $H^2(M, \mathcal{O}_M^*)$.

To define the Brauer group of M , consider all algebraic principal $\text{PGL}(r, \mathbb{C})$ -bundles on M for all $r \geq 1$. Let

$$\text{GL}(r, \mathbb{C}) \times \text{GL}(r', \mathbb{C}) \longrightarrow \text{GL}(rr', \mathbb{C})$$

be the homomorphism given by the natural action of any $A \times B \in \text{GL}(r, \mathbb{C}) \times \text{GL}(r', \mathbb{C})$ on $\mathbb{C}^r \otimes \mathbb{C}^{r'}$. This homomorphism descends to a homomorphism

$$\gamma : \text{PGL}(r, \mathbb{C}) \times \text{PGL}(r', \mathbb{C}) \longrightarrow \text{PGL}(rr', \mathbb{C}).$$

Given a principal $\text{PGL}(r, \mathbb{C})$ -bundle P and a principal $\text{PGL}(r', \mathbb{C})$ -bundle Q , consider the principal $\text{PGL}(r, \mathbb{C}) \times \text{PGL}(r', \mathbb{C})$ -bundle $P \times_M Q$ on M . From it, the above homomorphism γ produces a principal $\text{PGL}(rr', \mathbb{C})$ -bundle on M by extension of structure group. This principal $\text{PGL}(rr', \mathbb{C})$ -bundle will be denoted by $P \otimes Q$. A vector bundle V on M of rank r defines a principal $\text{GL}(r, \mathbb{C})$ -bundle, and the corresponding projective bundle $\mathbb{P}(V)$ defines a principal $\text{PGL}(r, \mathbb{C})$ -bundle. This principal $\text{PGL}(r, \mathbb{C})$ -bundle on M given by $\mathbb{P}(V)$ will also be denoted by $\mathbb{P}(V)$.

Two principal bundles P and Q are called *equivalent* if there are vector bundles V and W on M such that the principal bundle $P \otimes \mathbb{P}(V)$ is isomorphic to $Q \otimes \mathbb{P}(W)$. The equivalence classes of projective bundles form a group. The addition operation is given by the tensor product described above, and the inverse is given by the automorphism $A \mapsto (A^t)^{-1}$ of $\mathrm{PGL}(r, \mathbb{C})$ — it corresponds to sending a projective bundle to its dual projective bundle. The resulting group is called the *Brauer group* of M , and it is denoted by $\mathrm{Br}(M)$.

There is a natural injection $\mathrm{Br}(M) \rightarrow \mathrm{Br}'(M)$. This inclusion is known to be an isomorphism.

Let \mathcal{O}_M denote the sheaf of holomorphic functions on M . Consider the short exact sequence of sheaves on M

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \rightarrow 0,$$

where the homomorphism $\mathbb{Z} \rightarrow \mathcal{O}_M$ sends any integer n to the constant function $2\pi\sqrt{-1} \cdot n$. Let

$$(2.1) \quad \mathrm{Pic}(M) = H^1(M, \mathcal{O}_M^*) \xrightarrow{c} H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathcal{O}_M)$$

be the corresponding long exact sequence of cohomologies. The homomorphism c in (2.1) sends a holomorphic line bundle to its first Chern class. The image $c(\mathrm{Pic}(M))$ coincides with the Néron–Severi group

$$\mathrm{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z}).$$

Define the subgroup

$$(2.2) \quad A := H^2(M, \mathbb{Z})/c(\mathrm{Pic}(M)) = H^2(M, \mathbb{Z})/\mathrm{NS}(M) \subset H^2(M, \mathcal{O}_M)$$

(see (2.1)). Let

$$H^3(M, \mathbb{Z})_{\mathrm{tor}} \subset H^3(M, \mathbb{Z})$$

be the torsion part.

Proposition 2.1 ([Sc]). *There is a natural short exact sequence*

$$0 \rightarrow A \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \rightarrow \mathrm{Br}'(M) \rightarrow H^3(M, \mathbb{Z})_{\mathrm{tor}} \rightarrow 0,$$

where A is defined in (2.2).

See [Sc, p. 878. Proposition 1.1] for the proof of Proposition 2.1.

3. THE COHOMOLOGY OF SYMMETRIC PRODUCTS

Let X be an irreducible smooth complex projective curve. The genus of X will be denoted by g . For any positive integer d , let P_d be the group of permutations of $\{1, \dots, d\}$. By $\mathrm{Sym}^d(X)$ we will denote the quotient of X^d for the natural action of P_d on it. So $\mathrm{Sym}^d(X)$ parametrizes all formal sums of the form $\sum_{x \in X} n_x \cdot x$, where n_x are nonnegative integers with $\sum_{x \in X} n_x = d$. In other words, $\mathrm{Sym}^d(X)$ parametrizes all effective divisors

on X of degree d . This $\text{Sym}^d(X)$ is an irreducible smooth complex projective variety of complex dimension d . Let

$$(3.1) \quad q_d : X^d \longrightarrow \text{Sym}^d(X) = X^d/P_d$$

be the quotient map.

Let $\alpha_1, \alpha_2, \dots, \alpha_{2g}$ be a symplectic basis for $H^1(X, \mathbb{Z})$ chosen so that $\alpha_i \cdot \alpha_{i+g} = 1$ for $i \leq g$. The oriented generator of $H^2(X, \mathbb{Z})$ will be denoted by ω . For $i \in [1, 2g]$ and $j \in [1, d]$, we have the cohomology classes

$$(3.2) \quad \lambda_i^j := 1 \otimes \dots \otimes \alpha_i \otimes \dots \otimes 1 \in H^1(X^n, \mathbb{Z})$$

and

$$(3.3) \quad \eta^j := 1 \otimes \dots \otimes \omega \otimes \dots \otimes 1 \in H^2(X^n, \mathbb{Z}),$$

where α_i and ω are at the j -th position.

Theorem 3.1 ([Ma]). *For the morphism q_d in (3.1), the pullback homomorphism*

$$q_d^* : H^*(\text{Sym}^d(X), \mathbb{Z}) \longrightarrow H^*(X^d, \mathbb{Z})$$

is injective. Further, the image of q_d^ is generated, as a \mathbb{Z} -algebra, by*

$$\lambda_i = \sum_{j=1}^d \lambda_i^j, \quad 1 \leq i \leq 2g, \quad \text{and} \quad \eta = \sum_{j=1}^d \eta_j.$$

See [Ma, p. 325, (6.3)] and [Ma, p. 326, (7.1)] for Theorem 3.1.

There is a universal divisor D^{univ} on $\text{Sym}^d(X) \times X$; it consists of all $(z, x) \in \text{Sym}^d(X) \times X$ such that $x \in z$. We wish to describe the class of this divisor in $H^2(\text{Sym}^d(X) \times X, \mathbb{Z})$. In view of Theorem 3.1, the algebra $H^*(\text{Sym}^d(X) \times X, \mathbb{Z})$ is considered as a subalgebra of $H^*(X^{d+1}, \mathbb{Z})$.

For $i \in [1, d+1]$, let $\pi_i : X^{d+1} \longrightarrow X$ be the projection to the i -th factor. For any integer $k \in [1, d]$, consider the closed immersion

$$\iota_k : X^d \hookrightarrow X^{d+1}$$

uniquely determined by

$$\pi_i \circ \iota_k = \begin{cases} \pi'_i & \text{if } i \neq d+1 \\ \pi'_k & \text{if } i = d+1 \end{cases}$$

where π'_j is the projection of X^d to the j -th factor. The divisor on X^{d+1} given by the image of ι_k will be denoted by D_k . The classes

$$\lambda_i^j \cup \lambda_{i'}^{j'}, \quad i \neq i', \quad 1 \leq j < j' \leq d+1,$$

and

$$\eta^j, \quad 1 \leq j \leq d+1,$$

constructed as in (3.2) and (3.3) for $d+1$, together give a basis for $H^2(X^{d+1}, \mathbb{Z})$. We have the dual basis for $H^{2d}(X^{d+1}, \mathbb{Z})$ given by

$$\eta^{j^\vee} = \omega \otimes \dots \otimes \omega \otimes 1 \otimes \omega \otimes \dots \otimes \omega$$

and

$$\left(\lambda_i^j \cup \lambda_{i'}^{j'}\right)^\vee = \omega \otimes \cdots \otimes \omega \otimes \tilde{\alpha}_i \otimes \omega \otimes \cdots \otimes \omega \otimes \tilde{\alpha}_{i'} \otimes \omega \otimes \cdots \otimes \omega$$

where $\tilde{\alpha}_i$ (respectively, $\tilde{\alpha}_{i'}$) is the class with $\tilde{\alpha}_i \cup \alpha_i = \omega$ (respectively, $\tilde{\alpha}_{i'} \cup \alpha_{i'} = \omega$).
Now

$$\int_{D_k} \eta^{j^\vee} = \int_{X^d} \iota_\alpha^* \eta^{j^\vee} = \begin{cases} 1 & j = k \\ 1 & j = d+1 \\ 0 & \text{otherwise} \end{cases}$$

while

$$\int_{D_k} \left(\lambda_i^j \cup \lambda_{i'}^{j'}\right)^\vee = \int_X \tilde{\alpha}_i \cup \tilde{\alpha}_{i'}$$

if $j' = d+1$ and $j = k$, and

$$\int_{D_k} \left(\lambda_i^j \cup \lambda_{i'}^{j'}\right)^\vee = 0$$

otherwise. So the class of D_k is

$$\eta^k + \eta^{d+1} + \sum_{i=1}^g \lambda_i^k \cup \lambda_{i+g}^{d+1} - \sum_{i=g+1}^{2g} \lambda_i^k \cup \lambda_{i-g}^{d+1}.$$

By the Künneth formula, we have

$$H^2(\text{Sym}^d(X) \times X, \mathbb{Z}) = (H^2(\text{Sym}^d(X), \mathbb{Z}) \otimes H^0(X, \mathbb{Z}))$$

$$\oplus (H^0(\text{Sym}^d(X), \mathbb{Z}) \otimes H^2(X, \mathbb{Z})) \oplus (H^1(\text{Sym}^d(X), \mathbb{Z}) \otimes H^1(X, \mathbb{Z})).$$

Using (3.1), we have a basis for $H^2(\text{Sym}^d(X) \times X, \mathbb{Z})$ consisting of

$$\eta \otimes 1_X, (\lambda_i \cup \lambda_j) \otimes 1_X, 1_{\text{Sym}^d(X)} \otimes \omega, \lambda_i \otimes \alpha_j.$$

From the above computations it follows that the class of D^{univ} is

$$(3.4) \quad [D^{\text{univ}}] = \eta \otimes 1 + n(1_{\text{Sym}^d(X)} \otimes \omega) + \sum_{i=1}^g \lambda_i \otimes \alpha_{i+g} - \sum_{i=g+1}^{2g} \lambda_i \otimes \alpha_{i-g}.$$

Proposition 3.2.

(1) For a fixed point $x_0 \in X$, consider the inclusion

$$\iota_x : \text{Sym}^d(X) \hookrightarrow \text{Sym}^d(X) \times X$$

defined by $z \mapsto (z, x_0)$. The cohomology class $\iota_x^*[D^{\text{univ}}]$ is η .

(2) The slant product of $[D^{\text{univ}}]$ with α_i^\vee produces the class λ_i in $H^1(\text{Sym}^d(X), \mathbb{Z})$.

Proof. These follow from (3.4). □

4. BRAUER GROUP OF THE SYMMETRIC PRODUCT

Fix a point $x_0 \in X$. For any $d \geq 1$, let

$$(4.1) \quad f_d : \text{Sym}^d(X) \longrightarrow \text{Sym}^{d+1}(X)$$

be the morphism defined by $\sum_{x \in X} n_x \cdot x \longmapsto x_0 + \sum_{x \in X} n_x \cdot x$. Let

$$(4.2) \quad f_d^* : \text{Br}'(\text{Sym}^{d+1}(X)) \longrightarrow \text{Br}'(\text{Sym}^d(X))$$

be the pullback homomorphism for f_d in (4.1).

Lemma 4.1. *For every $d \geq 2$, the homomorphism f_d^* in (4.2) is an isomorphism.*

Proof. For every positive integer n , the cohomology group $H^*(\text{Sym}^n(X), \mathbb{Z})$ is torsionfree by Theorem 3.1. In particular,

$$H^3(\text{Sym}^n(X), \mathbb{Z})_{\text{tor}} = 0.$$

Therefore, from Proposition 2.1 we conclude that

$$(4.3) \quad \text{Br}'(\text{Sym}^n(X)) = (H^2(\text{Sym}^n(X), \mathbb{Z})/\text{NS}(\text{Sym}^n(X))) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}).$$

From Theorem 3.1,

$$(4.4) \quad H^2(\text{Sym}^n(X), \mathbb{Z}) = (\bigwedge^2 H^1(X, \mathbb{Z})) \oplus H^2(X, \mathbb{Z}).$$

Let

$$f'_d : H^2(\text{Sym}^{d+1}(X), \mathbb{Z}) \longrightarrow H^2(\text{Sym}^d(X), \mathbb{Z})$$

be the homomorphism that sends a cohomology class to its pullback by the map f_d in (4.1). It is evident that in terms of the isomorphism in (4.4), this homomorphism f'_d coincides with the identity map of $(\bigwedge^2 H^1(X, \mathbb{Z})) \oplus H^2(X, \mathbb{Z})$.

The isomorphism in (4.4) is clearly compatible with the Hodge type decompositions. Since f'_d coincides with the identity map of $(\bigwedge^2 H^1(X, \mathbb{Z})) \oplus H^2(X, \mathbb{Z})$, we now conclude that f'_d takes $\text{NS}(\text{Sym}^{d+1}(X))$ isomorphically to $\text{NS}(\text{Sym}^d(X))$. Therefore, the lemma follows from (4.3). \square

For any positive integer d , let

$$(4.5) \quad \xi_d : \text{Sym}^d(X) \longrightarrow \text{Pic}^d(X)$$

be the morphism defined by $\sum_{x \in X} n_x \cdot x \longmapsto \mathcal{O}_X(\sum_{x \in X} n_x \cdot x)$. Let

$$(4.6) \quad \xi_d^* : \text{Br}'(\text{Pic}^d(X)) \longrightarrow \text{Br}'(\text{Sym}^d(X))$$

be the pullback homomorphism corresponding to ξ_d .

Lemma 4.2. *For any $d \geq 2$, the homomorphism ξ_d^* in (4.6) is an isomorphism.*

Proof. Let $\eta_d : \text{Pic}^d(X) \rightarrow \text{Pic}^{d+1}(X)$ be the isomorphism defined by $L \mapsto L \otimes \mathcal{O}_X(x_0)$. We have the commutative diagram

$$\begin{array}{ccc} \text{Sym}^d(X) & \xrightarrow{f_d} & \text{Sym}^{d+1}(X) \\ \downarrow \xi_d & & \downarrow \xi_{d+1} \\ \text{Pic}^d(X) & \xrightarrow{\eta_d} & \text{Pic}^{d+1}(X) \end{array}$$

where f_d and ξ_d are constructed in (4.1) and (4.5) respectively, and η_d is defined above. Let

$$(4.7) \quad \begin{array}{ccc} \text{Br}'(\text{Pic}^{d+1}(X)) & \xrightarrow{\eta_d^*} & \text{Br}'(\text{Pic}^d(X)) \\ \downarrow \xi_{d+1}^* & & \downarrow \xi_d^* \\ \text{Br}'(\text{Sym}^{d+1}(X)) & \xrightarrow{f_d^*} & \text{Br}'(\text{Sym}^d(X)) \end{array}$$

be the corresponding commutative diagram of homomorphisms of cohomological Brauer groups. From Lemma 4.1 we know that f_d^* is an isomorphism for $d \geq 2$. The homomorphism η_d^* is an isomorphism because the map η_d is an isomorphism. Therefore, from the commutativity of (4.7) we conclude that the homomorphism ξ_d^* is an isomorphism if ξ_{d+1}^* is an isomorphism. Consequently, it suffices to prove the lemma for all d sufficiently large.

As before, the genus of X is denoted by g . Take any $d > 2g$. Note that for any line bundle L on X of degree d , using Serre duality we have

$$(4.8) \quad H^0(X, L) = H^0(X, K_X \otimes L^\vee)^\vee = 0$$

because $\text{degree}(K_X \otimes L^\vee) = 2g - 2 - d < 0$.

Take a Poincaré line bundle $\mathcal{L} \rightarrow X \times \text{Pic}^d(X)$. From (4.8) it follows that the direct image

$$\text{pr}_* \mathcal{L} \rightarrow \text{Pic}^d(X)$$

is locally free of rank $d - g + 1$, where pr is the projection of $X \times \text{Pic}^d(X)$ to $\text{Pic}^d(X)$. The projective bundle $\mathbb{P}(\text{pr}_* \mathcal{L})$, that parametrizes the lines in the fibers of the holomorphic vector bundle $\text{pr}_* \mathcal{L}$, is independent of the choice of the Poincaré line bundle \mathcal{L} . Indeed, this follows from the fact that any two choices of the Poincaré line bundle over $X \times \text{Pic}^d(X)$ differ by tensoring with a line bundle pulled back from $\text{Pic}^d(X)$ [ACGH, p. 166]. The total space of $\mathbb{P}(\text{pr}_* \mathcal{L})$ is identified with $\text{Sym}^d(X)$ by sending a section to the divisor on X given by the section. This identification between $\text{Sym}^d(X)$ and $\mathbb{P}(\text{pr}_* \mathcal{L})$ takes the map ξ_d in (4.5) to the natural projection of $\mathbb{P}(\text{pr}_* \mathcal{L})$ to $\text{Pic}^d(X)$.

Since $\mathbb{P}(\text{pr}_* \mathcal{L})$ is the projectivization of a vector bundle, the natural projection

$$\mathbb{P}(\text{pr}_* \mathcal{L}) \rightarrow \text{Pic}^d(X)$$

induces an isomorphism of cohomological Brauer groups [Ga, p. 193]. Consequently, the homomorphism

$$\xi_d^* : \text{Br}'(\text{Pic}^d(X)) \rightarrow \text{Br}'(\text{Sym}^d(X))$$

defined in (4.6) is an isomorphism if $d > 2g$. We noted earlier that it is enough to prove the lemma for all d sufficiently large. Therefore, the proof is now complete. \square

5. THE COHOMOLOGY OF THE QUOT SCHEME

For integers $r \geq 1$, d and m , denote by $\mathcal{Q}(r, d)$ the Quot scheme parametrizing all coherent subsheaves

$$\mathcal{F} \hookrightarrow \mathcal{O}_X^{\oplus r}$$

where \mathcal{F} is of rank r and degree $-d$. Note that there are no such subsheaf if $d < 0$. If $d = 0$, then $\mathcal{F} = \mathcal{O}_X^{\oplus r}$. If $d = 1$, then $\mathcal{Q}(r, d) = X \times \mathbb{CP}^{r-1}$. We assume that $d \geq 1$.

We will now recall from [Bi] a few facts about the Białynicki-Birula decomposition of $\mathcal{Q}(r, d)$. Using the natural action of $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ on \mathcal{O}_X , we get an action of \mathbb{G}_m^r on $\mathcal{O}_X^{\oplus r}$. This action produces an action of \mathbb{G}_m^r on $\mathcal{Q}(r, d)$. The fixed points of this torus action correspond to subsheaves of $\mathcal{O}_X^{\oplus r}$ that decompose into compatible direct sums

$$\bigoplus_{i=1}^r \mathcal{L}_i \hookrightarrow \mathcal{O}_X^{\oplus r},$$

where $\mathcal{L}_i \hookrightarrow \mathcal{O}_X$ is a subsheaf of rank one. Let D_i be the effective divisor given by the inclusion of \mathcal{L}_i in \mathcal{O}_X ; so $\mathcal{L}_i = \mathcal{O}_X(-D_i)$.

We use the convention that $\text{Sym}^0(X)$ is a single point. Using this notation,

$$(D_1, \dots, D_r) \in \text{Sym}^{m_1}(X) \times \dots \times \text{Sym}^{m_r}(X),$$

where $m_i = \text{degree}(D_i)$. Conversely, if $(D'_1, \dots, D'_r) \in \text{Sym}^{m_1}(X) \times \dots \times \text{Sym}^{m_r}(X)$, then the point of $\mathcal{Q}(r, d)$ representing the subsheaf

$$\bigoplus_{i=1}^r \mathcal{O}_X(-D'_i) \subset \mathcal{O}_X^{\oplus r}$$

is fixed by the action of \mathbb{G}_m^r .

For $k \geq 1$, denote by \mathbf{Part}_r^k the set of partitions of k of length r . So

$$\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{Part}_r^k$$

if and only if m_j are nonnegative integers with

$$\sum_{j=1}^r m_j = k.$$

For $\mathbf{m} \in \mathbf{Part}_r^d$, define

$$(5.1) \quad d_{\mathbf{m}} := \sum_{i=1}^r (i-1)m_i.$$

The connected components of the fixed point locus for the above action of \mathbb{G}_m^r on $\mathcal{Q}(r, d)$ are in bijection with the elements of \mathbf{Part}_r^d . The component corresponding to the partition $\mathbf{m} = (m_1, \dots, m_r)$ is the product

$$\text{Sym}^{\mathbf{m}}(X) := \text{Sym}^{m_1}(X) \times \dots \times \text{Sym}^{m_r}(X).$$

We can choose a 1-parameter subgroup $\mathbb{G}_m \rightarrow \mathbb{G}_m^n$ so that the fixed point loci remains unchanged if we restrict to the subgroup $\mathbb{G}_m \subset \mathbb{G}_m^r$ the above action. For the action on $\mathcal{Q}(r, d)$ of this subgroup \mathbb{G}_m , define

$$\mathrm{Sym}^{\mathbf{m}}(X)^+ := \{x \in \mathcal{Q}(r, d) \mid \lim_{t \rightarrow 0} t.x \in \mathrm{Sym}^{\mathbf{m}}(X)\},$$

where $\mathbf{m} \in \mathbf{Part}_r^k$. This stratification of $\mathcal{Q}(r, d)$ gives us a decomposition of the Poincaré polynomial of $\mathcal{Q}(r, d)$. Further, the morphism

$$(5.2) \quad \mathrm{Sym}^{\mathbf{m}}(X)^+ \longrightarrow \mathrm{Sym}^{\mathbf{m}}(X)$$

that sends a point to its limit is a fiber bundle with fiber $\mathbb{A}^{d_{\mathbf{m}}}$ [Bb], where $d_{\mathbf{m}}$ is defined in (5.1).

A further restriction on the above 1-parameter subgroup forces

$$(5.3) \quad \dim \mathrm{Sym}^{\mathbf{m}}(X)^+ = \dim \mathrm{Sym}^{\mathbf{m}}(X) + d_{\mathbf{m}} = d + d_{\mathbf{m}}$$

(see [Bi]).

Theorem 5.1. *For $i \geq 1$,*

$$H^i(\mathcal{Q}(r, d), \mathbb{Z}) = \bigoplus_{\substack{\mathbf{m} \in \mathbf{Part}_r^d \\ j+2d_{\mathbf{m}}=i}} H^j(\mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_r}(X), \mathbb{Z}).$$

Proof. See [Bi] and [BGL, p. 649, Remark]. □

We will construct some cohomology classes in $H^2(\mathcal{Q}(r, d), \mathbb{Z})$. There is a unique universal vector bundle $\mathcal{F}^{\mathrm{univ}}$ on $\mathcal{Q}(r, d) \times X$. Fix a point $x \in X$. Let

$$i_x : \mathcal{Q}(r, d) \longrightarrow \mathcal{Q}(r, d) \times X$$

be the embedding defined by $z \mapsto (z, x)$.

Let

$$(5.4) \quad c := i_x^* c_1(\mathcal{F}^{\mathrm{univ}}) \in H^2(\mathcal{Q}(r, d), \mathbb{Z})$$

be the pullback. This cohomology class c is clearly independent of x .

We can produce cohomology classes

$$\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{2g} \in H^1(\mathcal{Q}(r, d), \mathbb{Z})$$

by taking the slant product of $c_1(\mathcal{F}^{\mathrm{univ}})$ with the elements of a basis $\{\alpha_1, \dots, \alpha_{2g}\}$ for $H^1(X, \mathbb{Z})$. Finally, there is a cohomology class γ_2 obtained by taking the slant product of $c_2(\mathcal{F}^{\mathrm{univ}})$ with the fundamental class of X .

Proposition 5.2. *Suppose that $d \geq 2$. Then the classes*

$$c, \gamma_2, \bar{\alpha}_i \cup \bar{\alpha}_j, \quad 1 \leq i < j \leq 2g,$$

generate $H^2(\mathcal{Q}(r, d), \mathbb{Z})$. In fact, $H^2(\mathcal{Q}(r, d), \mathbb{Z})$ is torsionfree, and these classes form a basis of the \mathbb{Z} -module $H^2(\mathcal{Q}(r, d), \mathbb{Z})$.

Proof. Using Theorem 5.1 and Theorem 3.1 it follows that $H^2(\mathcal{Q}(r, d), \mathbb{Z})$ is torsionfree of rank

$$\binom{2g}{2} + 2.$$

Hence it suffices to show the stated classes generate the second cohomology group.

The torus action on $\mathcal{Q}(r, d)$ induces a Białynicki-Birula stratification on this variety, as described above. Using (5.3), the cell of largest dimension in the Białynicki-Birula decomposition is the cell corresponding to the partition

$$\mathbf{m}_1 = (d, 0, 0, \dots, 0),$$

and the second largest cell corresponds to the partition

$$\mathbf{m}_2 = (d-1, 1, 0, \dots, 0).$$

It follows that $\text{Sym}^{\mathbf{m}_1}(X)^+$ is an open sub-scheme of $\mathcal{Q}(r, d)$. Let $D := \mathcal{Q}(r, d) \setminus \text{Sym}^{\mathbf{m}_1}(X)^+$ be the complement. Using (5.2), we have

$$H^2(\text{Sym}^{\mathbf{m}_1}(X)^+, \mathbb{Z}) = H^2(\text{Sym}^{\mathbf{m}_1}(X), \mathbb{Z}).$$

Further, by a dimension calculation,

$$H^0(D, \mathbb{Z}) \cong H^0(\text{Sym}^{\mathbf{m}_2}(X), \mathbb{Z}).$$

Let

$$\iota : \text{Sym}^{\mathbf{m}_1}(X) \hookrightarrow \mathcal{Q}(r, d)$$

be the inclusion map.

The Gysin sequence for the decomposition $\mathcal{Q}(r, d) = \text{Sym}^{\mathbf{m}_1}(X)^+ \amalg D$ now reads:

$$\dots \longrightarrow H^0(\text{Sym}^{\mathbf{m}_2}(X), \mathbb{Z}) \xrightarrow{f_*} H^2(\mathcal{Q}(r, d), \mathbb{Z}) \xrightarrow{\iota^*} H^2(\text{Sym}^{\mathbf{m}_1}(X), \mathbb{Z}) \longrightarrow \dots,$$

where

$$(5.5) \quad f : \text{Sym}^{d-1}(X) \times X \longrightarrow \mathcal{Q}(r, d)$$

is the embedding. From [CS], this sequence splits. In other words, the Białynicki-Birula stratification, is integrally perfect.

To complete the proof, it suffices to verify the following two statements:

- (S1) The classes $\iota^*(\overline{\alpha}_i \cup \overline{\alpha}_j)$, $1 \leq i < j \leq 2g$, and $\iota^*(c)$ generate $H^2(\text{Sym}^{\mathbf{m}_1}(X), \mathbb{Z})$.
- (S2) The class γ_2 generates the image of f_* .

For (S1), observe that

$$\iota^*(\mathcal{F}^{\text{univ}}) = j_x^* \mathcal{O}_{\text{Sym}^d(X) \times X}(-D^{\text{univ}}) \oplus \mathcal{O}_{\text{Sym}^d(X)}^{r-1},$$

where $j_x : \text{Sym}^d(X) \longrightarrow \text{Sym}^d(X) \times X$ is the embedding defined by $x \longmapsto (x, z)$, and D^{univ} is the universal divisor on $\text{Sym}^d(X) \times X$. From, Proposition 3.2 and Theorem 3.1 it follows that the classes

$$\iota^*(c), \iota^*(\overline{\alpha}_i \cup \overline{\alpha}_j), \quad 1 \leq i < j \leq 2g,$$

give a basis for $H^2(\text{Sym}^{\mathbf{m}_1}(X), \mathbb{Z})$. Further, $\gamma_2 \in \text{kernel}(\iota^*)$.

For (S2), we assume that $r = 2$ for simplicity. The proof in the case of higher rank is obtained by adding some trivial summands to the argument below.

As noted above, we have a split exact sequence :

$$0 \longrightarrow H^0(\mathrm{Sym}^{d-1}(X) \times X, \mathbb{Z}) \xrightarrow{f_*} H^2(\mathcal{Q}(r, d), \mathbb{Z}) \xrightarrow{i^*} H^2(\mathrm{Sym}^d(X), \mathbb{Z}) \longrightarrow 0.$$

Fix some quotient

$$q : \mathcal{O}_X \longrightarrow Q \longrightarrow 0$$

of degree $d - 1$, and also fix some quotient

$$q' : \mathcal{O} \longrightarrow \mathcal{O}_p \longrightarrow 0$$

of degree 1, where $p \in X$ is a point not in the support of Q .

These gives us a point $z \in \mathrm{Sym}^{d-1}(X) \times X$. We can expand this to a morphism

$$F : \mathbb{A}^1 \longrightarrow (\mathrm{Sym}^{d-1}(X) \times X)^+$$

by considering the family of quotients

$$F(t) := \begin{pmatrix} qf & 0 \\ tq' & q' \end{pmatrix} : \mathcal{O}_X^{\oplus 2} \longrightarrow Q \oplus \mathcal{O}_p \longrightarrow 0.$$

Taking the closure of $F(\mathbb{A}^1)$ in $\mathcal{Q}(2, d)$ we obtain an inclusion

$$F : \mathbb{P}^1 \hookrightarrow \mathcal{Q}(2, d).$$

As $\dim \mathcal{Q}(2, d) = 2d$, this give a cohomology class

$$[\mathbb{P}^1] \in H^{4d-2}(\mathcal{Q}(2, d), \mathbb{Z}).$$

Let

$$\mathcal{W} \longrightarrow \mathbb{P}^1 \times X$$

be the restriction of the universal vector bundle $\mathcal{F}^{\mathrm{univ}} \longrightarrow \mathcal{Q}(2, d) \times X$. It fits in the short exact sequence

$$(5.6) \quad 0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times X}^{\oplus 2} \longrightarrow \tilde{Q} := (\mathcal{O}_{\mathbb{P}^1} \boxtimes Q) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_p \longrightarrow 0.$$

Note that the Chern character

$$(5.7) \quad Ch(\tilde{Q}) = d\omega_X + \omega_X \cup \omega_{\mathbb{P}^1},$$

where ω_X and $\omega_{\mathbb{P}^1}$ are the fundamental classes of X and \mathbb{P}^1 respectively. In particular, $c_1(\tilde{Q}) = \omega_X$. Therefore, the slant product of $c_1(\tilde{Q})$ with elements of $H^1(X, \mathbb{Z})$ vanish. We have

$$c_2(\tilde{Q}) = \omega_X \cup \omega_{\mathbb{P}^1}.$$

Its slant product with X is then just $\omega_{\mathbb{P}^1}$. Therefore,

$$(F^* \gamma_2) \cup [\mathbb{P}^1] = \int_{\mathbb{P}^1} \gamma_2 = 1.$$

So the cohomology classes described in the statement of the proposition give a basis for the vector space $H^2(\mathcal{Q}(r, d), \mathbb{Q})$.

We will prove the following statements:

$$(5.8) \quad (F^*c) \cup [\mathbb{P}^1] = 0.$$

$$(5.9) \quad \alpha_i \cup [\mathbb{P}^1] = 0.$$

$$(5.10) \quad f_*([\mathrm{Sym}^{d-1}(X) \times X]) \cup [\mathbb{P}^1] = 1.$$

$$(5.11) \quad (F^*\gamma_2) \cup [\mathbb{P}^1] = \int_{\mathbb{P}^1} \gamma_2 = 1.$$

The map f is defined in (5.5).

We first show that the above statements complete the proof. For that it is sufficient to observe that they imply both

$$f_*([\mathrm{Sym}^{d-1}(X) \times X]) \quad \text{and} \quad \gamma_2$$

are dual to $[\mathbb{P}^1]$ and hence must be equal.

To prove (5.8), consider (5.6). Choose a point $x \in X$ away from the support of $Q \oplus \mathcal{O}_p$ and restrict \mathcal{W} to $\mathbb{P}^1 \times \{x\}$. From (5.7) it follows that the first Chern class of this restriction vanish. The first Chern class of this restriction clearly coincides with $(F^*c) \cup [\mathbb{P}^1]$.

The left-hand side of (5.9) is clearly the slant product of $c_1(\mathcal{W})$ with α_i . We noted above that this slant product vanishes.

Now, (5.10) is clear from the construction of the morphism F from \mathbb{P}^1 . Finally (5.11) has already been proved. \square

6. THE BRAUER GROUP

For integers $r \geq 1$ and $d \geq 0$, let

$$(6.1) \quad \varphi : \mathcal{Q}(r, d) \longrightarrow \mathrm{Sym}^d(X)$$

be the morphism that sends any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ to the scheme theoretic support of the quotient $\mathcal{O}_X^{\oplus r}/\mathcal{F}$. Let

$$(6.2) \quad \varphi^* : \mathrm{Br}'(\mathrm{Sym}^d(X)) \longrightarrow \mathrm{Br}'(\mathcal{Q}(r, d))$$

be the pullback homomorphism using φ .

Lemma 6.1. *The homomorphism φ^* in (6.2) is an isomorphism.*

Proof. Note that $\mathrm{Br}'(\mathcal{Q}(r, d)) = \mathrm{Br}'(\mathrm{Sym}^d(X)) = 0$ if $d \leq 1$. Therefore, we assume that $d \geq 2$.

The cohomology group $H^3(\mathcal{Q}(r, d), \mathbb{Z})$ is torsionfree. Indeed, this follows from Theorem 5.1 and the fact that $H^*(\mathrm{Sym}^n(X), \mathbb{Z})$ is torsionfree [Ma, p. 329, (12.3)]. Therefore, Proposition 2.1 says that

$$(6.3) \quad \mathrm{Br}'(\mathcal{Q}(r, d)) = (H^2(\mathcal{Q}(r, d), \mathbb{Z})/\mathrm{NS}(\mathcal{Q}(r, d))) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}).$$

Let

$$\varphi' : H^2(\mathrm{Sym}^d(X), \mathbb{Z}) \longrightarrow H^2(\mathcal{Q}(r, d), \mathbb{Z})$$

be the pullback homomorphism using φ in (6.1). Recall from Theorem 3.1 the description of $H^2(\mathrm{Sym}^d(X), \mathbb{Z})$. From Proposition 5.2 we conclude that φ' is injective, and

$$(6.4) \quad H^2(\mathcal{Q}(r, d), \mathbb{Z}) = \mathrm{image}(\varphi') \oplus \mathbb{Z} \cdot \gamma_2,$$

where γ_2 is the cohomology class in Proposition 5.2. Take any point $y := (y_1, \dots, y_d) \in \mathrm{Sym}^d(X)$ such that all y_i are distinct. Then $\varphi^{-1}(y)$ is a product of copies of \mathbb{CP}^{r-1} , hence $H^1(\varphi^{-1}(y), \mathbb{Z}) = 0$. From this it follows that the image of the cup product

$$H^1(\mathcal{Q}(r, d), \mathbb{Z}) \otimes H^1(\mathcal{Q}(r, d), \mathbb{Z}) \longrightarrow H^2(\mathcal{Q}(r, d), \mathbb{Z})$$

is in the image φ' . If the point $x \in X$ in (5.4) is different from all y_i , then the restriction of the universal vector bundle $\mathcal{F}^{\mathrm{univ}}$ (see (5.4)) to $\varphi^{-1}(y)$ is the trivial vector bundle of rank r . From this it follows that c is in the image of φ' .

From (6.4) it follows immediately that

$$\mathrm{NS}(\mathcal{Q}(r, d)) = \varphi'(\mathrm{NS}(\mathrm{Sym}^{-d}(X))) \oplus \mathbb{Z} \cdot \gamma_2.$$

In view of (6.3), from this we conclude that φ^* in (6.2) is an isomorphism if $d \geq 2$. \square

As before, fix a point $x_0 \in X$. Let

$$\delta : \mathcal{Q}(r, d) \longrightarrow \mathcal{Q}(r, d+r)$$

be the morphism that sends any $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ represented by a point of $\mathcal{Q}(r, d)$ to the point representing the subsheaf $\mathcal{F} \otimes \mathcal{O}_X(-x_0) \subset \mathcal{O}_X^{\oplus r}$. Let

$$(6.5) \quad \delta^* : \mathrm{Br}'(\mathcal{Q}(r, d+r)) \longrightarrow \mathrm{Br}'(\mathcal{Q}(r, d))$$

be the pullback homomorphism by δ .

Corollary 6.2. *For any $d \geq 2$, the homomorphism δ^* in (6.5) is an isomorphism.*

Proof. As in (6.1), define

$$\psi : \mathcal{Q}(r, d+r) \longrightarrow \mathrm{Sym}^{d+r}(X)$$

to be the morphism that sends any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ to the scheme theoretic support of the quotient $\mathcal{O}_X^{\oplus r}/\mathcal{F}$. Let

$$h : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Sym}^{d+r}(X)$$

be the morphism defined by $\sum_{x \in X} n_x \cdot x \longmapsto r \cdot x_0 + \sum_{x \in X} n_x \cdot x$. The following diagram of morphisms

$$\begin{array}{ccc} \mathcal{Q}(r, d) & \xrightarrow{\delta} & \mathcal{Q}(r, d+r) \\ \downarrow \varphi & & \downarrow \psi \\ \mathrm{Sym}^d(X) & \xrightarrow{h} & \mathrm{Sym}^{d+r}(X) \end{array}$$

is commutative, where φ is defined in (6.1). Consider the corresponding commutative diagram

$$\begin{array}{ccc} \mathrm{Br}'(\mathrm{Sym}^{d+r}(X)) & \xrightarrow{h^*} & \mathrm{Br}'(\mathrm{Sym}^d(X)) \\ \downarrow \psi^* & & \downarrow \varphi^* \\ \mathrm{Br}'(\mathcal{Q}(r, d+r)) & \xrightarrow{\delta^*} & \mathrm{Br}'(\mathcal{Q}(r, d)) \end{array}$$

of homomorphisms. If $d \geq 2$, from Lemma 6.1 we know that ψ^* and φ^* are isomorphisms, while Lemma 4.1 implies that h^* is an isomorphism. Therefore, δ^* is an isomorphism. \square

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